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TABLES FOR THE PIDDUCK-KENT SPECIAL SOLUTION
FOR THE MOTION OF THE POWDER GAS IN A GUN

by

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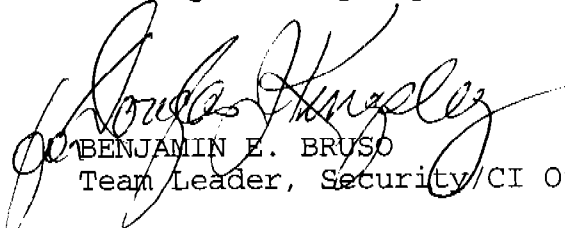
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Abstract

A discussion is first given of the general problem of the flow of gas in a gun, of the simplified problem known as the Lagrange problem, and of the Pidduck-Kent special solution of the differential equation of the Lagrange problem. This special solution is commonly supposed to be the limiting solution of the latter. An improved derivation is then given of the Pidduck-Kent solution, the improvement consisting in accounting for gas imperfection by means of a covolume correction.

The Pidduck-Kent solution is then put into a form suitable for easy calculation. To summarize the results, let the following symbols denote certain quantities as calculated from the Pidduck-Kent solution, viz

- p_o = breech pressure
- p_b = pressure at the base of the projectile
- \bar{p} = space-mean pressure
- W_p = kinetic energy of the projectile
- W_c = kinetic energy of the powder

Then, according to the Pidduck-Kent solution,

$$\frac{\bar{p}}{p_b} = \frac{W_c + W_p}{W_p} = 1 + \epsilon/\delta$$

$$\frac{p_o}{p_b} = (1 - a_o)^{-n-1}$$

$$1/a_o = \frac{2n+3}{\delta} + \frac{2(n+1)}{\epsilon}$$

where ϵ denotes the charge-projectile mass ratio c/m , n the polytropic index $1/(\gamma-1)$, (where γ is the effective ratio of specific heats that takes heat loss into account), and a_o a parameter characteristic of the Pidduck-Kent solution.

The quantity δ is given by

$$\frac{1}{\delta} = \frac{1}{2n+3} \left[1 + \alpha n \left(\frac{1+c_1 \rho n}{1+c_1 n} \right) \right]$$

The quantity α , depending only on ϵ , may be found by linear interpolation in Table I of Appendix A. The quantity ρ , also depending only on ϵ , may be found by linear interpolation in Table II of Appendix A. The quantity c_1 , depending weakly on both ϵ and n , may be found by two-way linear interpolation in Table III of Appendix A or by reading from the graphs or the contour map in Appendix B. Ordinarily reading from the graphs is the easiest and most accurate of the three methods of finding c_1 . With these aids one may expect to obtain Pidduck-Kent values (as distinguished of course from true values) with the following maximum errors for the range $\epsilon = 0$ to 10, $\gamma = 1.2$ to 3:

$\frac{1}{\delta}$ 1 PART IN 5000

Q_0 1 PART IN 6700

P_0/P_b 1 PART IN 3700

$\bar{P}/P_b = \frac{W_c + W_p}{W_p}$ 1 PART IN 5000

Section I. Elementary Corrections for the Motion of the Powder Gas

In cases where [REDACTED] the initial mass c of the powder is small compared to the mass m of the projectile, it is customary to correct for the motion of the powder gas by means of certain very simple formulas. These are:

$$P_0 = P_b (1 + \epsilon/2) \quad (1)$$

$$\bar{P} = P_b (1 + \epsilon/3) \quad (2)$$

$$W_c = (\epsilon/3) (\frac{1}{2} m V^2) \quad (3)$$

$$\text{where } \epsilon \equiv c/m \quad (4)$$

and p_0 denotes the pressure at the breech, p_b the pressure at the base of the projectile, \bar{p} the space-mean pressure behind the projectile, V the velocity of the projectile, and W_c the kinetic energy of the powder plus powder gas. These formulas are derived on the assumptions that the cross-sectional area is uniform all the way from the breech to the base of the projectile and that the powder grains and powder gas together form a fluid of uniform density. Under these conditions the velocity of this fluid varies linearly from the value zero at the breech to the value V at the base of the projectile. The correction for the kinetic energy of the charge, viz addition of one-third the mass of the charge to that of the projectile, is then similar to that for the kinetic energy of a spring (without waves) to which a bob is attached.

Section II. The General Problem of the Motion of the Powder.

Actually the problem is somewhat more complicated than the simple theory would indicate. Gas friction is not taken into account at all in the above formulas for the pressure ratios. Chambrage, the drop in cross-sectional area on going from the powder chamber to the bore, introduces a non-uniformity into the cross-sectional area, a non-uniformity which will probably become much more serious as one goes in the direction of guns of much higher velocity. The powder gas is compressible, so that there actually must be a variation of density with position as well as time. It is not really correct to lump the powder grains and the powder gas together as a single fluid. Finally, the burning of the powder furnishes energy to the gas, so that the expansion is not adiabatic.

Section III. The Lagrange Problem.

When Lagrange set himself the problem of solving for the motion of the powder gas in a gun, he was able to formulate it in a relatively simple fashion, because the propellant of his day was simply loose black powder. In such a case it is not too bad to assume instantaneous combustion. He therefore assumed that he had a cylinder, closed at one end by the breech and at the other, variable, end by the base of the projectile and filled initially with hot gas of uniform pressure, density, and temperature. His problem was then to calculate the subsequent states of the system, the projectile being initially at rest.

Lagrange's problem has received its most complete treatment by Love and Pidduck*, who applied their results to the calculation of p_o/p_b and W_c/W_p versus travel for the case of a 150 mm. gun. (W_p denotes the kinetic energy of the projectile.) Pidduck found that these ratios oscillated but seemed to approach certain limiting values, which values corresponded to a certain special solution of the differential equation of motion of the gas. The rigorous solution is of a wave character, involving rarefaction waves travelling back and forth between breech and projectile, and satisfies the initial conditions of the Lagrange problem. The special solution, on the other hand, is of a non-wave character and does not satisfy the initial conditions of the Lagrange problem, but corresponds to an initial non-uniform distribution of pressure and density. From observation of the computational results for p_o/p_b Pidduck and all later investigators have suspected but not proved that the accurate solution approaches the special solution in the limit of large travel.

Solution IV. The Pidduck-Kent Special Solution.

The special solution has also been derived by Kent** and applied by Hirschfelder et al*** to the computation of the ratios \bar{p}/p_b and W_c/W_p . In the limiting case of sufficiently small values of ϵ , the Pidduck-Kent special solution gives values for the above ratios that agree with the uniform-density values of equations (2) and (3). The deviations from the uniform-density values become important in the case of large values of ϵ , i.e. for high-velocity guns. As we have said above, the effect of gas

*Love and Pidduck, Phil. Trans. Roy. Soc. 222, 167 (1922)

**R.H. Kent, Physics 7, 319 (1936)

***Hirschfelder, Kershner, and Curtiss, NDRC Report A-142
Hirschfelder, Kershner, and Sherman, NDRC Report A-204

friction also becomes more important for high velocities, as does the effect of chambrage. The latter becomes more important because the use of a high c/m ratio necessitates the use of a powder chamber much fatter than the bore. For small values of ϵ , for which the assumption of uniform density may be made, the effect of chambrage may be estimated by the methods of BRL Reports No. 307* and No. 3**. For large values of ϵ it is doubtful that those methods would apply.

Although the solution of the hydrodynamical problem thus involves a number of complications when we go to large values of ϵ , it appears worth while to isolate the effect of non-uniformity of gas density by publishing adequate tables of the Pidduck-Kent special solution. These tables may not eventually be useful for practical calculations, but they should be of considerable help in the analysis of experimental results on high-velocity guns, if only to tell us, by comparison of measured pressure distributions with the Pidduck-Kent solution, how important are the neglected factors.

Section V. An Improved Derivation of the Pidduck-Kent Special Solution

In the Lagrange problem, of which this is a special solution of the corresponding differential equation***, it is assumed that the chamber and bore form a cylinder of uniform cross-section, the powder being all burned initially. We follow the notation of Kent in the above-mentioned paper.

Thus

$x \equiv$ initial distance of any gas particle from the breech

$y \equiv$ distance at time t of the same gas particle from the breech

$\rho \equiv$ density of the gas

$p \equiv$ pressure of the gas

The subscript zero denotes initial values. A subscript on a partial derivative denotes the variable which is kept constant in the differentiation. The equation of continuity is then

$$\rho = \rho_0 \left(\frac{\partial y}{\partial x} \right)_t^{-1} \quad (5)$$

*J.P. Vinti, BRL Report No. 307, "The Equations of Interior Ballistics", Chap. IV.

**N.F. Ramsey, Jr. (with J.R. Lane), BRL Report No. 3, "Analysis of Pressure-Time Curves for the Gerlich Rifle"

***It is not a solution of the actual Lagrange problem, because the initial conditions are not satisfied.

The equation of motion is

$$\rho \left(\frac{\partial^2 y}{\partial t^2} \right)_x = - \left(\frac{\partial P}{\partial y} \right)_t \quad (6)$$

The equation of the state of the gas is

$$P \left(\frac{1}{\rho} - \eta \right) = R_1 T \quad (7)$$

where η denotes the specific covolume of the powder gas, T the absolute temperature, and R_1 the gas constant per unit mass. If

$$\gamma' \equiv C_p / C_v \quad (8)$$

denotes the ratio of specific heats of the gas, the equation of an adiabatic is

$$P \left(\frac{1}{\rho} - \eta \right)^{\gamma'} = P_0 \left(\frac{1}{\rho_0} - \eta \right)^{\gamma'} \quad (9)$$

Kent further assumes that "the original pressure and density distribution follows adiabatically from an antecedent regime of uniform pressure and density." Under these circumstances (9) becomes

$$P \left(\frac{1}{\rho} - \eta \right)^{\gamma'} = P_0 \left(\frac{1}{\rho_0} - \eta \right)^{\gamma'} = A \text{ CONSTANT } K$$

The derivation follows through equally well if we replace the true γ' by a larger effective value γ for the ratio of specific heats, a device that affords a means of taking heat loss into account.* Furthermore it is not necessary to take the right side of (10) to be a constant; it suffices for the following derivation to take it as $k(\psi(t))$,** where $\psi(t)$ is a function of time alone. Kent suggests that by such a means one can allow partially for the burning of the powder. It is doubtful, however, that any such simple artifice will be of use in attempting to treat the case where the powder is still burning. Indeed the continuity equation (5) and the dynamical equation (6) do not hold in that case, unless the density ρ be taken as the density of a "fluid" composed of powder gas and solid powder together, in which case the above relations between pressure and density do not hold.

* See J. P. Vinti, BRL Report No. 307, "The Equations of Interior Ballistics"

J. P. Vinti, BRL Report No. 402, "Project for a New Table for Interior Ballistics for Multiperforated Powder."

J. P. Vinti and Jack Chernick, BRL Report No. 625, "Interior Ballistics for Powder of Constant Burning Surface."

**Suggested by Kent, loc. cit.

The derivation of the special solution for an imperfect gas has been given partially by Kent, who showed that the solution is the same as that for a perfect gas through powers of $\epsilon \equiv c/m$ as high as the third. In this section we show that this restriction can be removed, the formulas obtained being entirely independent of the value of η . Pidduck* has outlined a proof, but we believe it desirable to put the complete derivation on record.

Following Kent, we introduce the variable

$$z \equiv y - \eta \int_0^x \rho_0(x) dx \quad (11)$$

and attempt to find a special solution of (6) in the form

$$z = f(x) \phi(t) \quad (12)$$

At $t = 0$ we have $y = x$ so that

$$f(x) \phi(0) = x - \eta \int_0^x \rho_0(x) dx \quad (13)$$

From (12) it is clear that either f or ϕ may contain an arbitrary factor, so that we are free to choose the function ϕ in such a way that $\phi(0) = 1$. Then

$$f(x) = x - \eta \int_0^x \rho_0(x) dx \quad (14)$$

From (5), (11), and (12) we find

$$\rho_0 \left[\frac{1}{\rho} - \eta \right] = f'(x) \phi(t) \quad (15)$$

Introducing the new variable

* F. B. Pidduck, Journal of Applied Physics 8, 144 (1937)

$$W \equiv \left(\frac{1}{\rho} - \eta\right)^{-1} \quad (16)$$

we have $\frac{\rho_0}{f'} = W\phi \quad (17)$

where f' is short for $f'(x)$.

Now $f'(x) = 1 - \eta\rho_0(x) = \rho_0\left(\frac{1}{\rho_0} - \eta\right),$

so that $\frac{\rho_0}{f'} = \left(\frac{1}{\rho_0} - \eta\right)^{-1} = W_0 = W\phi \quad (18)$

Making use of the fact that

$$\left(\frac{\partial^2 y}{\partial t^2}\right)_x = \left(\frac{\partial^2 z}{\partial t^2}\right)_x,$$

WE HAVE FROM (6) $\rho\left(\frac{\partial^2 z}{\partial t^2}\right)_x = -\left(\frac{\partial p}{\partial y}\right)_t = -\left(\frac{\partial p}{\partial x}\right)_t \left(\frac{\partial y}{\partial x}\right)_t^{-1} \quad (19)$

Using the continuity equation (5), we then find

$$\rho_0\left(\frac{\partial^2 z}{\partial t^2}\right)_x = -\left(\frac{\partial p}{\partial x}\right)_t \quad (20)$$

On writing $\left(\frac{\partial p}{\partial x}\right)_t$ as $\left(\frac{\partial p}{\partial f}\right)_t f'$ and using (18), we find

$$W_0\left(\frac{\partial^2 z}{\partial t^2}\right)_f = \left(\frac{\partial p}{\partial f}\right)_t \quad (21)$$

IN TERMS OF w , (10) BECOMES

$$p = K w^{\gamma} \psi(t) = K w_0^{\gamma} \phi^{-\gamma} \psi(t) \quad (22)$$

FROM (12), (21), AND (22)

$$w_0 f \phi''(t) = -K \gamma w_0^{\gamma-1} \phi^{-\gamma} \psi \frac{dw_0}{df} \quad (23)$$

SEPARATING VARIABLES, WE OBTAIN

$$\frac{\phi'' \phi^{\gamma}}{\psi} = \frac{-\gamma K w_0^{\gamma-2}}{f} \frac{dw_0}{df} = B, \text{ A CONSTANT} \quad (24)$$

INTEGRATION OF (24), WITH USE OF THE CONDITIONS $f=0$

AND $w_0 = w_0(0)$ AT $x=0$, GIVES

$$w_0^{\gamma-1} = w_0^{\gamma-1}(0) - \frac{B(\gamma-1) f^2}{2K\gamma} \quad (25)$$

$$\text{OR} \quad w_0(f) = w_0(0) \left(1 - a f^2\right)^{\frac{1}{\gamma-1}} \quad (26)$$

$$\text{WHERE} \quad a \equiv \frac{B(\gamma-1)}{2K\gamma} w_0^{1-\gamma}(0) \quad (27)$$

Let x_b denote the initial distance from the breech to the base of the projectile, A the uniform cross-sectional area, and abbreviate $f(x_b)$ by f_b . Then, using the fact that when the powder is all burned the total mass of the gas is equal to the original powder mass c , we have

$$C = A \int_0^{x_b} \rho_0(x) dx = A \int_0^{f_b} \rho_0 \frac{dx}{df} df = A \int_0^{f_b} \frac{\rho_0}{f} df = A \int_0^{f_b} w_0 df \quad (28)$$

If we now use (26) and (28), introduce the dummy variable μ defined by $f = \mu f_b$, and use the abbreviations

$$a_0 \equiv a f_b^2 \quad (29)$$

$$S \equiv \int_0^1 (1 - a_0 \mu^2)^{1/(\gamma-1)} d\mu \quad (30)$$

we find

$$c = A w_0(0) f_b S \quad (31)$$

At the base of the projectile we have from (12) and (22)

$$\left(\frac{\partial^2 z}{\partial t^2} \right)_{x_b} = f_b \phi''(t) \quad (32)$$

$$P_b = K w_0^\gamma(f_b) \phi^{-\gamma} \psi(t) \quad (33)$$

If m denotes the mass of the projectile (increased perhaps by some factor a little greater than unity if one wants to allow for bore friction), the equation of motion of the projectile then becomes

$$f_b \phi'' = \frac{A}{m} P_b = \frac{AK}{m} w_0^\gamma(f_b) \phi^{-\gamma} \psi \quad (34)$$

Then

$$\frac{\phi'' \phi^\gamma}{\psi} = \frac{AK w_0^\gamma(f_b)}{m f_b} = B \quad (35)$$

where the constant B is the same as in (24) and (27). Comparing (35) with (27), with the use of (26) and (29), we find

$$A w_0(0) f_b (1 - a_0)^{\frac{\gamma}{\gamma-1}} = \frac{2\gamma}{\gamma-1} a_0 m \quad (36)$$

Dividing (36) by (31), we then find

$$\epsilon = \frac{2\gamma}{\gamma-1} a_0 (1 - a_0)^{-\gamma/(\gamma-1)} S \quad (37)$$

If γ and ϵ are given, the parameter a_0 may be calculated from (37),

The Ratio P_0/P_b

From (22), (26), and (29) the ratio of breech pressure p_0 to base pressure p_b is given by

$$P_0/P_b = (1 - a_0)^{-\gamma/(\gamma-1)} \quad (38)$$

The Kinetic Energy W_p of the Powder Gas:

Since the gas velocity $(\partial y / \partial t)_x = (\partial z / \partial t)_x$, we have

$$W_p = \frac{A}{2} \int_0^{y_b} \rho \left(\frac{\partial z}{\partial t} \right)^2 dy = \frac{A}{2} \int_0^{x_b} \rho \left(\frac{\partial z}{\partial t} \right)^2 \left(\frac{\partial y}{\partial x} \right)_t dx \quad (39)$$

PUTTING $dx = df/f'$, $\partial z / \partial t = f \phi'(t)$ FROM (12), $\rho \partial y / \partial x = \rho_0$ FROM (5), AND $\rho_0 / f' = W_0$ FROM (18), WE OBTAIN

$$W_p = \frac{A}{2} \phi'^2 \int_0^{f_b} W_0 f^2 df \quad (40)$$

At the base of the projectile the gas velocity $f_b \phi'$ equals the projectile velocity V . Use this fact, insert (26) into (40), replace f by μf_b , use (29), and replace $A W_0(0) f_b$ BY c/S ACCORDING TO (31). ONE FINDS

$$W_p = \frac{1}{2} \frac{c}{S} \frac{V^2}{f_b^2} \int_0^1 (1 - a_0 \mu^2)^{1/(x-1)} \mu^2 d\mu \quad (41)$$

The Ratio \bar{P}/P_b

We next investigate the ratio of the mean pressure \bar{P} to the base pressure P_b , where \bar{P} denotes the mean pressure that should be used in the equation of state. In interior ballistics the equation of state is used to eliminate the temperature, which occurs in the expression for the internal energy of the gas. To write down the total internal energy we need to find a mass average for the specific internal energy $c_v T$. Now

$$c_v T = \frac{c_v}{R_1} P \left(\frac{1}{\rho} - \eta \right) \quad (42)$$

The total internal energy E_i is then given by

$$E_i = \frac{c_v}{R_1} \int_0^{y_b} P \left(\frac{1}{\rho} - \eta \right) A \rho dy \quad (43)$$

The value used for ρ in interior ballistic calculations is the average value

$$\rho_{AV} = \frac{c}{A y_b} \quad (44)$$

The appropriate mean pressure \bar{P} for use in the equation of state is then given by

$$\frac{c c_v}{R_1} \bar{P} \left(\frac{1}{\rho_{AV}} - \eta \right) = \frac{c_v}{R_1} \int_0^{y_b} P \left(\frac{1}{\rho} - \eta \right) A \rho dy \quad (45)$$

NOW FROM (11) AND (12) WE HAVE

$$f_b \phi = y_b - \eta \int_0^{x_b} \rho_0(x) dx = y_b - \eta c/A \quad (46)$$

THEN

$$\frac{1}{\rho_{av}} - \eta = \frac{A y_b}{c} - \eta = \frac{A}{c} \left(y_b - \frac{\eta c}{A} \right) = \frac{A}{c} f_b \phi \quad (47)$$

FROM (45) AND (47)

$$\bar{p} f_b \phi = \int_0^{y_b} p \left[\frac{1}{\rho} - \eta \right] \rho dy \quad (48)$$

NOW PUT $dy = \frac{\partial y}{\partial x} dx$, $\rho \frac{\partial y}{\partial x} = \rho_0$, $dx = df/f'$, $\frac{\rho_0}{f'} = w_0$, $\frac{1}{\rho} - \eta = w^{-1}$, AND USE (22). THE RESULT IS

$$\bar{p} f_b \phi = K \phi^{1-\gamma} \psi \int_0^{f_b} w_0^\gamma df \quad (49)$$

NOW USE (26) AND PUT $f = \mu f_b$. THE RESULT IS

$$\bar{p} = K \phi^{-\gamma} \psi w_0^\gamma(0) \int_0^1 (1 - a_0 \mu^2)^{\gamma/(\gamma-1)} d\mu \quad (50)$$

NOW WRITE DOWN (22) FOR p_b , DIVIDE \bar{p} BY p_b , EXPRESS $\frac{w_0(0)}{w_0(f_b)}$ BY MEANS OF (26) AND (29), AND REPLACE THE EXPRESSION THAT OCCURS, VIZ. $(1 - a_0)^{-\gamma/(\gamma-1)}$, BY $\frac{\gamma-1}{2\gamma} \frac{\epsilon}{a_0 S}$ ACCORDING TO (37). THE RESULT IS

$$\frac{\bar{p}}{p_b} = \frac{\gamma-1}{2\gamma} \frac{\epsilon}{a_0 S} \int_0^1 (1 - a_0 \mu^2)^{\gamma/(\gamma-1)} d\mu \quad (51)$$

THE INTEGRAL MAY BE INTEGRATED BY PARTS TO GIVE

$$\int_0^1 (1 - a_0 \mu^2)^{\frac{\gamma}{\gamma-1}} d\mu = (1 - a_0)^{\frac{\gamma}{\gamma-1}} + \frac{2\gamma}{\gamma-1} a_0 \int_0^1 (1 - a_0 \mu^2)^{\frac{1}{\gamma-1}} \mu^2 d\mu \quad (52)$$

THEN, FROM (51), (52), AND (37)

$$\frac{\bar{p}}{p_b} = 1 + \frac{\epsilon}{\delta} \quad (53)$$

WHERE

$$\frac{1}{\delta} \equiv \frac{1}{S} \int_0^1 (1 - a_0 \mu^2)^{\frac{1}{\gamma-1}} \mu^2 d\mu \quad (54)$$

The kinetic energy of the powder gas (41) may also be expressed in terms of δ :

$$W_p = \frac{1}{2} c V^2 / \delta, \quad (55)$$

so that the total kinetic energy of projectile and powder can be expressed as

$$W_{TOTAL} = \frac{1}{2} m V^2 (1 + \frac{\epsilon}{\delta}) \quad (56)$$

From (56) we see that the total kinetic energy of projectile and powder can be expressed as

$$W_{TOTAL} = (\frac{1}{2}) m_e V^2 \quad (57)$$

where the effective mass m_e is given by

$$m_e = m (1 + \frac{\epsilon}{\delta}) \quad (58)$$

Note that the correct mean pressure \bar{p} for use in the equation of state is equal to the effective pressure p_e that has to be assumed to act on the effective mass m_e to impart to it the total kinetic energy $(\frac{1}{2}) m_e V^2$. To show this note that $A p_e = m_e dv/dt$ gives the correct total kinetic energy (on integration of $A p_e v dt$). Dividing this expression by $A p_b = m dv/dt$ we find $p_e/p_b = 1 + \epsilon/\delta = \bar{p}/p_b$, as was to be shown.

Section VI: Discussion of the Above Results

Letting*

$$n \equiv \frac{1}{\delta - 1} \quad (59)$$

we have from (30), (37), and (54)

$$S = \int_0^1 (1 - a_0 \mu^2)^n d\mu \quad (60)$$

$$\epsilon = 2(n+1) a_0 (1 - a_0)^{-n-1} S \quad (61)$$

$$1/\delta = \frac{1}{S} \int_0^1 (1 - a_0 \mu^2)^n \mu^2 d\mu \quad (62)$$

In applications δ and ϵ are given and a_0 and $1/\delta$ are required. We first show that the integral in (62) can be reduced to an expression involving S . Let

$$R \equiv \int_0^1 (1 - a_0 \mu^2)^n \mu^2 d\mu \quad (63)$$

* n is called the "polytropic index" in astrophysics.

By expressing $\mu^2 d\mu$ as $\frac{1}{2} \mu d\mu^2$ we can express the integral as

$$R = -\frac{1}{2(n+1)a_0} \int_{\mu=0}^1 \mu d(1-a_0\mu^2)^{n+1},$$

which can be integrated by parts to give

$$R = -\frac{(1-a_0)^{n+1}}{2(n+1)a_0} + \frac{1}{2(n+1)a_0} \int_0^1 (1-a_0\mu^2)^n (1-a_0\mu^2) d\mu,$$

from which

$$2(n+1)a_0 R = -(1-a_0)^{n+1} + S - a_0 R$$

Solution for R and division by S gives

$$\frac{1}{\delta} \equiv \frac{R}{S} = \frac{1}{(2n+3)a_0} \left[1 - \frac{(1-a_0)^{n+1}}{S} \right]$$

comparison of which with (61) gives

$$\frac{1}{\delta} = \frac{1}{2n+3} \left[\frac{1}{a_0} - \frac{2(n+1)}{\epsilon} \right] \quad (64)$$

From (38) and (59) we have also

$$P_0/P_b = (1-a_0)^{-n-1}$$

From (64) and (65) it is clear that the calculation of all the practical results of the solution has now been reduced to the calculation of a_0 from the given values of ϵ and n . We consider this calculation and its tabulation in Section VII.

To set limits on the parameter a_0 we proceed as follows.

From (26) and (29) or from (38) we see first that a_0 must be real.

Now in (38) we have $\gamma/(\gamma-1) = n+1$ and we know that $n \equiv 1/(\gamma-1)$ is a real positive number, since $\gamma > 1$. Thus if $a_0 < 0$

we have $P_0/P_b < 1$, an absurd result, since there must be a positive pressure difference $P_0 - P_b > 0$ to overcome the

inertia of the powder gas. If $a_0 > 1$, $P_0/P_b = (1-a_0)^{-n-1}$ may be

positive real only if n is an odd integer, any other value of n giving either a complex or a negative result. We may therefore rule out the case $a_0 > 1$ as non-physical, since any slight change of n from an odd integer value must lead to a non-physical result.

Thus $0 \leq a_0 \leq 1$; examination of (60) and (61) shows that $a_0 = 0$ corresponds to $\epsilon = 0$ and $a_0 = 1$ to $\epsilon = \infty$. For very

small ϵ we have, from (60) and (61), $a_0 = \epsilon/[2(n+1)]$. We obtain the second approximation by binomial expansion of the integrand in S and of the expression $(1-a_0)^{-n-1}$ in powers of a_0 , thereby obtaining $(1-a_0)^{-n-1} S = 1 + (1+2n/3)a_0$, and then replacing this a_0 by its first approximation. This second approximation is

$$a_0 = \frac{\epsilon}{2(n+1)} \left[1 - \frac{2n+3}{6(n+1)} \epsilon \right],$$

insertion of which into (64) gives $\delta = 3$ for $\epsilon \rightarrow 0$, in agreement with (2) and (3). Insertion of the first approximation for a_0 into (65), which may be written

$$P_0/P_b = (1-a_0)^{-n-1} = 1 + (n+1)a_0 + \dots$$

GIVES

$$P_0/P_b = 1 + \frac{\epsilon}{2} \text{ FOR } \epsilon \rightarrow 0, \text{ IN AGREEMENT WITH (1).}$$

Section VII. The Methods of Calculation and Tabulation

Our problem now is to calculate a_0 as a function of n and ϵ from (60) and (61) and then to devise tables by means of which a_0 and $1/\delta$ can easily be found when n and ϵ are given. To do so we might calculate, for a given value of n , the quantity ϵ for a number of evenly spaced values of a_0 and then inverse interpolate. In this way we could construct a table of a_0 versus ϵ and n for equally spaced values of both. In doing so we should have to evaluate the integral S , which can be expressed as an incomplete beta-function, tables of which are available, by Karl Pearson. (We did, in fact, make preliminary computations in this way.)

In order to explain most easily our subsequent calculations it is desirable at this point to discuss the tabulation possibilities. We wish to construct tables with ϵ and n as arguments that will enable the user to calculate both a_0 and δ . In these double-entry tables it is desirable that linear interpolation be permissible with respect to both arguments ϵ and n . It seems pointless, however, to tabulate both a_0 and δ , since the user of the tables can find either one from the other by (64), undoubtedly more quickly than by interpolation in an extra double-entry table. If the table is of a_0 , it is clear from (64) that in calculating δ the user will lose accuracy by subtraction. From a table of δ , however, he will be able to calculate a_0 without loss of accuracy.

It is settled, therefore, that we must tabulate δ or some simple function thereof. The functions that come to mind on inspection of (64) are $1/\delta$ and $(2n+3)/\delta$. Since we can probably insure the permissibility of linear interpolation with respect to ϵ only by taking a rather small interval for the latter (increasing, as we see later, with increase in ϵ), our problem is to find which of the three functions δ , $1/\delta$, or $(2n+3)/\delta$ will give the best interpolation possibilities for n . That is, for fixed ϵ , which of these three functions is most nearly a straight line when plotted against n ? The above mentioned preliminary computations show unmistakably

that only $(2n+3)/\delta$ is reasonably straight.

We showed above that $\delta=3$ for $\epsilon=0$ and that $a_0=1$ corresponds to $\epsilon \rightarrow \infty$, so that by (64), $\delta \rightarrow 2n+3$ for $\epsilon \rightarrow \infty$. The resulting simple table of limiting values

ϵ	δ	$1/\delta$	$(2n+3)/\delta$
0	3	$1/3$	$1 + 2n/3$
∞	$2n+3$	$1/(2n+3)$	1

then shows that the curve of $1/\delta$ versus n is not expected to be straight for large values of ϵ . To see why the curve of $(2n+3)/\delta$ versus n is expected to be straighter than that of δ versus n for all ϵ we next examine the behavior of the function $(2n+3)/\delta$ in the limiting cases of small n and large n . This investigation leads us to the function that we actually tabulate.

To do so, we first rewrite (60) and (61) in terms of the quantity

$$b \equiv \frac{a_0}{1-a_0} \quad (66)$$

From (60), (61), and (66) we then have

$$\epsilon = 2(n+1)b \int_0^1 [1+b(1-\mu^2)]^n d\mu \quad (67)$$

Since $0 \leq a_0 \leq 1$, we have $b \geq 0$, $b=0$ corresponding to $\epsilon=0$ and $b=\infty$ to $\epsilon=\infty$. To investigate the case of small n we write the integrand of (67) as $\exp\{n \ln[1+b(1-\mu^2)]\}$ and expand this exponential in a power series in n . The result is

$$\epsilon = 2(n+1)b \left[1 + n \int_0^1 \ln[1+b(1-\mu^2)] d\mu + \dots \right]$$

Now as n approaches zero we see that b approaches the value $\epsilon/2$. Any deviation of b from the value $\epsilon/2$ in the logarithmic term will result in a quadratic (n^2) contribution to the expression in brackets. Thus, through linear terms only

$$\frac{\epsilon}{2(n+1)b} = 1 + nJ + \dots \quad (68)$$

$$\text{WHERE } J \equiv \int_0^1 \ln \left[1 + \frac{\epsilon}{2}(1-\mu^2) \right] d\mu$$

$$= \left(1 + \frac{2}{\epsilon}\right)^{1/2} \ln \left[1 + \epsilon + \{\epsilon(2+\epsilon)\}^{1/2} \right] - 2 \quad (69)$$

The evaluation of this integral is given in Appendix C. For sufficiently small ϵ it has the value $\epsilon/3$. From (64) and (66) we have

$$\frac{2n+3}{\delta} = 1 + \frac{1}{b} - \frac{2(n+1)}{\epsilon} \quad (70)$$

Comparing (68) and (70), we obtain through terms linear in n

$$\frac{2n+3}{\delta} = 1 + \frac{2Jn}{\epsilon} + \dots, \quad (71)$$

from which

$$\delta = 3 + (2 - \frac{6J}{\epsilon})n + \dots \quad (72)$$

The initial slopes are thus $\frac{2J}{\epsilon}$ for $\frac{2n+3}{\delta}$ versus n and $2 - \frac{6J}{\epsilon}$ for δ versus n .

To investigate the case of large n we first note in (67) that since $b \geq 0$ the integrand and thus the integral are both equal to or greater than unity. Thus from (67) as $n \rightarrow \infty$ for a fixed value of ϵ , we must have $b \rightarrow 0$. Since the integrand then takes on the indeterminate form 1^∞ , we rewrite it as before as $\exp\{n \ln[1 + b(1-\mu^2)]\}$, which becomes $\exp[nb(1-\mu^2)]$

for the limiting case $b \rightarrow 0$. For $n \rightarrow \infty$ we may also replace the outside factor $(n+1)$ by n , thereby obtaining

$$\epsilon = 2nb \int_0^1 \exp[nb(1-\mu^2)] d\mu \quad (73)$$

$$\text{SO THAT } b = K(\epsilon)/n, \quad (74)$$

WHERE $K(\epsilon)$ IS A SOLUTION OF

$$2Ke^K \int_0^1 e^{-K\mu^2} d\mu = \epsilon \quad (75)$$

Then from (70) and (74)

$$\frac{2n+3}{\delta} = 1 + \frac{n}{K(\epsilon)} - \frac{2(n+1)}{\epsilon} \quad (76)$$

which gives for $n \rightarrow \infty$

$$\frac{1}{\delta} = \frac{1}{2K(\epsilon)} - \frac{1}{\epsilon} \quad (77)$$

It is an interesting check at this point to note that $n \rightarrow \infty$ corresponds to $\gamma \rightarrow 1$, i.e. to an expansion process governed by $P_0(1/\rho - \eta)\psi(t) = P(1/\rho - \eta)$, which process is

isothermal if $\psi(t)$ is constant. The equation of motion (20) can then easily be integrated directly, with (75) and (77) as direct results.

On putting $K\mu^2 \equiv \xi^2$ in (74), we obtain

$$\epsilon = \pi^{1/2} \sigma H(\sigma) \exp(\sigma^2), \quad (78.1)$$

WHERE $\sigma \equiv K^{1/2}$

$$\text{AND } H(\sigma) \equiv 2(\pi^{-1/2}) \int_0^\sigma e^{-\xi^2} d\xi \quad (78.2)$$

of which there is a very good table.* With the aid of this table we may calculate the values of ϵ for evenly spaced values of σ and, by inverse interpolation, then obtain the values of σ and of $K \equiv \sigma^2$ for evenly spaced values of ϵ .

From (77) the final slope of δ versus n for large n is clearly zero, while from (76) the final slope of $(2n+3)/\delta$ versus n is $(1/K - 2/\epsilon)$. We may now summarize the initial and final behavior of the curves δ versus n and $(2n+3)/\delta$ versus n .

	δ VERSUS n	$\frac{(2n+3)}{\delta}$ VERSUS n
INITIAL SLOPE	$2 - 6J/\epsilon$	$2J/\epsilon$
FINAL SLOPE	0	$1/K - 2/\epsilon$

Now $J \approx \epsilon/3$ for sufficiently small ϵ and $J \approx \ln \epsilon$ for sufficiently large ϵ , so that the initial slope of δ versus n varies with increasing ϵ from 0 to 2 while the final slope is always zero. Actual plots indeed show marked curvature for the curves of δ versus n .

The following table gives initial and final slopes for the curves $(2n+3)/\delta$ versus n , for a few values of ϵ .

*"Tables of Probability Functions", Vol. I, by the WPA, sponsored by the National Bureau of Standards (1941)

Slopes of the Curves $(2n+3)/\delta$ Versus n :

ϵ	Initial Slope α	Final Slope $\alpha\beta$	Difference
0	0.6667	0.6667	0.0000
2	0.4929	0.5602	0.0673
4	0.4038	0.5086	0.1048
6	0.3471	0.4756	0.1285
8	0.3070	0.4515	0.1445
10	0.2768	0.4329	0.1561
∞	0.0000	0.0000	0.0000

At $\epsilon = 10$ the difference of the slopes is clearly approaching a maximum value which will probably be less than 0.2. Since the tables of the present report end at $\epsilon = 10$, we have not seen fit to extend the above table beyond that value. In any case it appears that the curves of $(2n+3)/\delta$ versus n are very much straighter than those of δ versus n .

Further investigation then showed that the quantity $(2n+3)/\delta$ can be expressed as follows in terms of a very weakly varying function $C, (\epsilon, n)$.

$$\frac{2n+3}{\delta} = 1 + \alpha n \left[\frac{1 + C, \beta n}{1 + C, n} \right] \quad (79)$$

where α is the slope of $(2n+3)/\delta$ versus n for $n=0$ and $\alpha\beta$ is its slope for $n=\infty$. Thus

$$\alpha = \frac{2J}{\epsilon} \quad (80)$$

WHERE

$$J = \sqrt{1 + \frac{2}{\epsilon}} \ln [1 + \epsilon + \sqrt{\epsilon(2 + \epsilon)}] - 2 \quad (69)$$

$$\text{AND } \alpha\beta = \frac{1}{K(\epsilon)} - \frac{2}{\epsilon}, \quad (81)$$

where $K(\epsilon)$ is given by (78). Over the ranges $0 \leq \epsilon \leq 10$, $1/2 \leq n \leq 5$, the quantity c , differs very little from unity, ranging from 0.960 to 1.065. Such behavior is not surprising, for a constant value of c , will clearly reproduce the correct behavior of $(2n+3)/8$ for very small n and for very large n ; it gives the values $1 + (2/\epsilon)J$ for very small n , in agreement with (71) and $1 + \frac{n}{K(\epsilon)} - \frac{2n}{\epsilon}$ for very large n , in agreement with (76). Indeed replacement of c , by unity in (79) gives values of $1/8$ not worse than one part in 1000 for any of the values of ϵ and n that we consider. Such accuracy would probably meet the practical requirements of interior ballistics, but it appears worth while to obtain somewhat better accuracy, since this Pidduck-Kent solution has considerable mathematical interest as the limiting solution of a classical problem in fluid mechanics.

From (70) and (79)

$$c_1 = \frac{Q-1}{(\beta-Q)n} \quad (82)$$

WHERE

$$Q \equiv \frac{1}{2n} \left[\frac{1}{b} - \frac{2(n+1)}{\epsilon} \right] \quad (83)$$

The final c_1 -table of this report has a range on n from 0.5 to 5, corresponding to values of the effective γ ranging from the value 1.2 (approximately the value for ballistite, but uncorrected for heat loss) up to the value 3, which would allow for enormous heat loss.

We found b by solution of (67), which, for an integer value of n , gives ϵ as a polynomial in b . For $n=1$ we had simply to solve a quadratic equation. For $n=2, 3, 4$, and 5 we tabulated ϵ versus b , at equal intervals, and inverse interpolated to find b versus ϵ at equal intervals. In obtaining a double-entry table of c_1 versus ϵ and n we found the following argument intervals satisfactory for linear interpolation:

$$\epsilon = 0(0.2)1(1)10 \text{ AND } n = 1/2, 1, 2, 3, 4, 5$$

To obtain reasonable smoothness in this table we found it necessary, because of the subtractions in (82) and (83), to calculate b to high accuracy*, especially for the smaller values of ϵ . This circumstance, conversely, accounts for the fact that very few significant figures in the c_1 -table will give $1/8$ to high accuracy.

* This necessity rendered useless the Pearson tables of the incomplete beta-function.

For $n = 1/2$ we had to calculate, according to (67), values of the integral

$$\begin{aligned} I_{1/2} &\equiv \int_0^1 [1 + b(1-\mu^2)]^{1/2} d\mu \\ &= [1/2] [1 + (1+b)b^{-1/2} \tan^{-1} b^{1/2}] \end{aligned} \quad (84)**$$

In obtaining tables of b and c , in this case we found it convenient first to tabulate ϵ versus $b^{1/2}$, at equal intervals of $b^{1/2}$, and then to inverse interpolate for the latter rather than for b .

We did not have to compute b or c , for any other values of n , except for the purpose of checking, which we did very simply as follows. The integral $I_n \equiv \int_0^1 [1 + b(1-\mu^2)]^n d\mu$ in (67) satisfies the recursion formula

$$I_{n+1} = \frac{1}{2n+3} [2(n+1)(1+b)I_n + 1] \quad (85)**$$

We chose several values of b and for each of these values calculated I_n for $n = 1/2, 3/2, 5/2, 7/2$, AND $9/2$ by means of (84) and successive applications of (85). Then by (67) we easily computed the value of ϵ for each of these cases and, from (70), the accurate value of $1/\delta$. To check the adequacy of the unit n -interval of the c -table we first interpolated linearly in the latter (or used the graphs or contour map given later). We then interpolated in the tables of α and β to find the values of the latter quantities corresponding to each ϵ and then inserted the values of n, α, β , and c , into (79) to find $1/\delta$. Comparisons with the accurate values of $1/\delta$ showed that the error was never greater than that arising from the use of tabular values of n . (See Section VIII for a discussion of errors of the tables.)

For the table of $\alpha(\epsilon)$ we computed anchor values directly from the formula

$$\alpha = \frac{2}{\epsilon} \left[\left(1 + \frac{2}{\epsilon}\right)^{1/2} \ln [1 + \epsilon + \sqrt{\epsilon(2+\epsilon)}] - 2 \right], \quad (86)$$

** For the derivation see Appendix C.

deducible directly from (69) and (80). These anchor values were $\epsilon = 0(0.1)2.6(0.2)5.0(0.5)10.0$. Direct interpolation then gave values of α for the final tabular arguments: $\epsilon = 0(0.05)1.0(0.1)4.0(0.2)10.0$.

For the table of $\beta(\epsilon)$ we first used (78), with the aid of the WPA "Tables of Probability Functions", to make a table of ϵ for $\sigma = 0.18(0.02)1.30$. Inverse interpolation then gave tables of σ and thus of $\alpha\beta = \frac{1}{\sigma^2} - \frac{2}{\epsilon}$ for the same anchor values of ϵ as for α . Division gave a table of β for those same anchor values and direct interpolation then gave the table of β for the final tabular arguments: $\epsilon = 0(0.1)2.6(0.2)5.0(0.25)10.0$.

Section VIII. Discussion of the Tables and their Accuracy

Appendix A gives the final working tables for $\alpha(\epsilon)$, $\beta(\epsilon)$, and $c, (\epsilon, n)$. Appendix B gives a set of graphs of c , versus ϵ for $n = \frac{1}{2}, 1, 2, 3, 4, 5$, a contour map of c , as a function of ϵ and n , and a table of values* read from the graphs and used in constructing the contour map.

The values of α and β are given to four decimal places and are correct in all cases to four decimal places, the original anchor values of α and $\alpha\beta$ having been calculated to seven decimal places. The values of c , are given to three decimal places, rounded down from four decimal places. The possible error in a tabular value of α or of β is thus not greater than 0.5 in the fourth decimal, i.e. 0.00005. Examination of the differences shows that linear interpolation is permissible throughout either table. With a maximum possible error of absolute value 0.00005 in either of two neighboring entries, it is clear that the permissible linear interpolation cannot give an error of magnitude greater than 0.00005 either for α or for β . Experience with the c , table (which is not subdivided quite finely enough to make linear interpolation always rigorously permissible) and with the c , graphs and contour map shows that the possible error is one unit in the third decimal place, i.e. 0.001.

We now ask the question: how much error can be produced in $\frac{1}{s}$ by these possible errors $\Delta\alpha = 0.00005$, $\Delta\beta = 0.00005$, AND $\Delta c = 0.001$? We have

$$\frac{2n+3}{s} = 1 + \alpha n \left[\frac{1 + c, \beta n}{1 + c, n} \right] \quad (79)$$

* This table is included to facilitate possible reproduction of this report in other forms.

A word of warning is appropriate here. In evaluating the fraction $(1+c_1\beta n)/(1+c_1n)$ it would be entirely incorrect to keep only three decimals in the numerator and three in the denominator, simply because c_1 is given only to three decimals. One can best see this point by considering the case $\epsilon=2, n=1$ for which $\beta=1.1364$ and $c_1=1.061$. We compute the fraction $(1+c_1\beta n)/(1+c_1n)$ as 1.070 for $c_1=1.061$ and 1.068 for $c_1=1$. Thus rounding c_1 to zero decimal places leaves unaffected the second decimal place of the fraction, so that we are justified in keeping two more decimal places in the result than were used for c_1 . Instead of using elaborate rules for the proper number of significant figures, however, one will probably find it simpler to use more places than are necessary or valid in calculating (79) and simply round down at the end to the proper number of figures for $1/\delta$ that are indicated by the considerations about to be given.

Error in $1/\delta$ from Error in α Alone

Let $\Delta_\alpha(1/\delta)$ be the error in $1/\delta$ arising from insertion into (79) of a value of α in error by $\Delta\alpha$. From (79) we have

$$\Delta_\alpha\left(\frac{1}{\delta}\right) = \left(\frac{1+c_1\beta n}{1+c_1n}\right)\left(\frac{n}{2n+3}\right)\Delta\alpha \quad (87)$$

Placing $c_1=1$ and $\beta=1.564$ (its value for $\epsilon=10$) we find that the coefficient has its largest value for $n=5$ (for the range of values of n that occur in the tables). This value being 0.565, we find for the largest possible error in $1/\delta$ that can arise from the possible error 0.00005 in α , the value

$$\Delta_{\alpha \text{ MAX}}(1/\delta) = 0.00003,$$

or three units in the fifth decimal place of $1/\delta$.

Error in $1/\delta$ from Error in β Alone

Let $\Delta_\beta(1/\delta)$ be the error in $1/\delta$ arising from insertion into (79) of a value of β in error by $\Delta\beta$. Placing $c_1=1$ in (79) we then have

$$\Delta_\beta\left(\frac{1}{\delta}\right) = \frac{\alpha n^2}{(1+n)(2n+3)} \Delta\beta \quad (88)$$

The largest value of $n^2(1+n)^{-1}(2n+3)^{-1}$ for our range of values of n occurs at $n=5$ and has the value 0.32. The largest value of α is $2/3$ for $\epsilon=0$. For a possible error $\Delta\beta=0.00005$ we thus find

$$\Delta_{\beta \text{ MAX}}(1/\delta) = (0.32)\left(\frac{2}{3}\right)(0.00005) = 0.00001,$$

or one unit in the fifth decimal place of $1/\delta$.

Error in $1/\delta$ from Error in c , Alone

Let $\Delta_c(1/\delta)$ be the error in $1/\delta$ arising from insertion into (79) of a value of c , in error by Δc . From (79) we have

$$\Delta_c(1/\delta) = \frac{\alpha(\beta-1)n^2}{(1+n)^2(2n+3)} \Delta c, \quad (89)$$

on differentiating (79) and putting $c \approx 1$. The largest value of $\alpha(\beta-1)$ is about 0.16, occurring for $\epsilon=10$, and the largest value for the function $n^2(1+n)^{-2}(2n+3)^{-1}$ is about 0.064, occurring at $n \approx$

2.3. On insertion of these values and the maximum possible value $\Delta c = 0.001$, we find

$$\Delta_{c, \max}(1/\delta) = 0.00001$$

or one unit in the fifth decimal place of $1/\delta$.

Thus in the practical use of Eq. (79) and the tables for α, β , and c , we may expect a maximum possible error of $3+1+1=5$ units in the fifth decimal place of $1/\delta$. Now the smallest $1/\delta$ for our ranges of ϵ and n , occurring for $\epsilon=10$ and $n=5$, is about 0.23, so that the maximum possible error is about 5 parts in 23,000 or 1 part in 4600. Roughly, then, the maximum possible error is about 1 part in 5000 in calculating $1/\delta$ by (79) and the tables. It would be quite feasible to reduce this possible error to a much lower value by using finer intervals and more decimals in the tables of α and β and still retain the desirable feature of permissible linear interpolation. With the use of such bulkier tables of α and β one could reduce the possible error to that arising from error in c , alone, or about 1 part in 25,000. We have not felt it worth while to do so, however.

Possible Error in a_0

From (64) we have

$$\frac{1}{a_0} = \frac{2n+3}{\delta} + \frac{2(n+1)}{\epsilon} \quad (90)$$

so that

$$\Delta a_0 = -a_0^2(2n+3)\Delta\left(\frac{1}{\delta}\right) \quad (91)$$

From (90) and (91)

$$\frac{\Delta a_0}{a_0} = -\left[\frac{2n+3}{\delta} + \frac{2(n+1)}{\epsilon}\right]\Delta\left(\frac{1}{\delta}\right) \quad (92)$$

We have seen that by far the largest part of the possible error in $1/\delta$ arises from the possible error in α and that the largest possible error in $1/\delta$ due to error in α occurs at $n=5$. To obtain an upper limit to $|\Delta a_0/a_0|$ we therefore insert into (92) the value $n=5$ and, obviously, the largest value of ϵ , viz. 10, that occurs in our tables. Using the value $(2n+3)/\delta = 3.034$ for $\epsilon = 10$ and $n=5$ and the value $|\Delta(1/\delta)|_{\text{MAX}} = 0.00005$ we find

$$|\Delta a_0/a_0| = 0.00015$$

The maximum possible error in a_0 is thus only 1.5 parts in 10,000 or 1 part in 6700. It is clear that the maximum possible absolute error also corresponds to the same values of ϵ and n , for which $a_0 = 0.2363$, so that $|\Delta a_0|_{\text{MAX}} = 0.000035$.

Possible Error in P_0/P_b

The Pidduck-Kent value of the ratio of breech pressure to base pressure is given by

$$\frac{P_0}{P_b} = (1-a_0)^{-n-1} \quad (65)$$

Thus

$$\Delta(P_0/P_b) = (n+1)(1-a_0)^{-n-2} \Delta a_0$$

and

$$\frac{\Delta(P_0/P_b)}{(P_0/P_b)} = \frac{(n+1)\Delta a_0}{1-a_0} \quad (93)$$

On inserting the values $\Delta a_0 = 0.000035$, $a_0 = 0.2362$, and $n = 5$, we find

$$\text{MAX} \left| \frac{\Delta(P_0/P_b)}{(P_0/P_b)} \right| = 0.00027$$

The maximum possible error that can arise in calculating the Pidduck-Kent value for this pressure ratio from our tables is thus 2.7 parts in 10,000 or 1 part in 3700.

Possible Error in \bar{P}/P_b OR W_c/W_p

The Pidduck-Kent value for the ratio of space mean pressure to base pressure, or for the ratio of the total kinetic energy of the powder plus projectile to that of the projectile, is given by

$$\bar{P}/P_b = \frac{W_c + W_p}{W_p} = 1 + \epsilon/\delta$$

The greatest possible error in either ratio is thus ϵ times the greatest possible error in $1/8$ or 0.00005ϵ . For $\epsilon = 10$ and $n = 5$ the fractional error thus becomes

$$\frac{0.0005}{1 + (0.23)10} = 0.0002$$

The greatest possible error that can arise in calculating the Pidduck-Kent value for this ratio from our tables is thus 1 part in 5000.

John P. Vinti
John P. Vinti

Sidney Kravitz
Sidney Kravitz

APPENDIX A

TABLES FOR THE PIDDUCK-KENT SOLUTION

$$\frac{1}{\delta} = \frac{1}{2n+3} \left\{ 1 + \alpha n \left(\frac{1+c, \beta n}{1+c, n} \right) \right\}$$

where $n = 1/(\gamma - 1)$ and α, β , and c , are given in the tables which follow below.

TABLE I

In the following table α is given as a function of ϵ for the following values: $\epsilon = 0(.05)1(.1)4(.2)10$. It is found from the following formula: $\alpha = \frac{2}{\epsilon} \left[\sqrt{1 + \frac{\epsilon}{2}} \left\{ \ln(1 + \epsilon + \epsilon \sqrt{1 + \frac{\epsilon}{2}}) \right\} - 2 \right]$

The first forward differences of the function are given in the third column marked Δ_1 . Linear interpolation is permissible.

ϵ	α	Δ_1	ϵ	α	Δ_1	ϵ	α	Δ_1
.00	.6667	-66	1.0	.5621	-81	4.0	.4038	-67
.05	.6601	-64	1.1	.5540	-78	4.2	.3971	-65
.10	.6537	-62	1.2	.5462	-75	4.4	.3906	-62
.15	.6475	-61	1.3	.5387	-73	4.6	.3844	-59
.20	.6414	-59	1.4	.5314	-70	4.8	.3785	-57
.25	.6355+	-57	1.5	.5244	-67	5.0	.3728	-56
.30	.6298	-56	1.6	.5177	-65	5.2	.3672	-53
.35	.6242	-55	1.7	.5112	-63	5.4	.3619	-51
.40	.6187	-53	1.8	.5049	-61	5.6	.3568	-49
.45	.6134	-52	1.9	.4988	-59	5.8	.3519	-48
.50	.6082	-51	2.0	.4929	-57	6.0	.3471	-46
.55	.6031	-50	2.1	.4872	-56	6.2	.3425+	-44
.60	.5981	-48	2.2	.4816	-53	6.4	.3381	-43
.65	.5933	-48	2.3	.4763	-53	6.6	.3338	-42
.70	.5885+	-46	2.4	.4710	-50	6.8	.3296	-41
.75	.5839	-46	2.5	.4660	-50	7.0	.3255+	-39
.80	.5793	-44	2.6	.4610	-48	7.2	.3216	-38
.85	.5749	-44	2.7	.4562	-46	7.4	.3178	-37
.90	.5705+	-42	2.8	.4516	-46	7.6	.3141	-36
.95	.5663	-42	2.9	.4470	-44	7.8	.3105+	-35
			3.0	.4426	-43	8.0	.3070	-34
			3.1	.4383	-42	8.2	.3036	-33
			3.2	.4341	-41	8.4	.3003	-32
			3.3	.4300	-40	8.6	.2971	-31
			3.4	.4260	-39	8.8	.2940	-31
			3.5	.4221	-39	9.0	.2909	-30
			3.6	.4182	-37	9.2	.2879	-29
			3.7	.4145+	-36	9.4	.2850+	-28
			3.8	.4109	-36	9.6	.2822	-27
			3.9	.4073	-35	9.8	.2795-	-27
						10.0	.2768	

TABLE II

In the following table β is given as a function of ϵ for the following values: $\epsilon = 0(.1)2.6(.2)5(.25)10$.

$$\beta = \frac{1}{2} \left[\frac{1}{k} - \frac{2}{\epsilon} \right] \text{ where } k \text{ is the solution of } \epsilon = 2ke^k \int_0^1 e^{-k\mu^2} d\mu$$

The first forward differences of the function are given in the column marked Δ_1 . Linear interpolation is permissible.

ϵ	β	Δ_1	ϵ	β	Δ_1	ϵ	β	Δ_1
0.0	1.0000	67	2.5	1.1686	63	5.00	1.3160	136
0.1	1.0067	69	2.6	1.1749	125	5.25	1.3296	135
0.2	1.0136	69	---	-----	---	5.50	1.3431	134
0.3	1.0205-	69	2.8	1.1874	124	5.75	1.3565+	132
0.4	1.0274	70						
0.5	1.0344	69	3.0	1.1998	122	6.00	1.3697	131
0.6	1.0413	70	3.2	1.2120	121	6.25	1.3828	129
0.7	1.0483	70	3.4	1.2241	119	6.50	1.3957	128
0.8	1.0553	69	3.6	1.2360	118	6.75	1.4085-	126
0.9	1.0622	69	3.8	1.2478	117			
1.0	1.0691	69	4.0	1.2595-	115	7.00	1.4211	126
1.1	1.0760	69	4.2	1.2710	114	7.25	1.4337	124
1.2	1.0829	68	4.4	1.2824	113	7.50	1.4461	123
1.3	1.0897	68	4.6	1.2937	112	7.75	1.4584	121
1.4	1.0965+	68	4.8	1.3049	111			
1.5	1.1033	67				8.00	1.4705+	121
1.6	1.1100	67				8.25	1.4826	120
1.7	1.1167	66				8.50	1.4946	118
1.8	1.1233	66				8.75	1.5064	118
1.9	1.1299	65						
2.0	1.1364	66				9.00	1.5182	116
2.1	1.1430	64				9.25	1.5298	116
2.2	1.1494	65				9.50	1.5414	115
2.3	1.1559	64				9.75	1.5529	113
2.4	1.1623	63				10.00	1.5642	

TABLE III

In the following table c_1 is given as a function of ϵ and n for all combinations of the following values: $\epsilon = 0(.2)1(1)10$; $n = 1/2, 1(1)5$.

ϵ	n					
	1/2	1	2	3	4	5
0.0	1.000	1.000	1.000	1.000	1.000	1.000
.2	1.016	1.016	1.016	1.016	1.016	1.016
.4	1.029	1.029	1.029	1.029	1.029	1.030
.6	1.038	1.039	1.039	1.039	1.039	1.039
.8	1.045	1.046	1.046	1.047	1.047	1.047
1	1.051	1.051	1.052	1.052	1.053	1.053
2	1.059	1.061	1.063	1.064	1.065	1.065
3	1.053	1.057	1.061	1.063	1.064	1.065
4	1.042	1.047	1.053	1.055	1.057	1.058
5	1.029	1.036	1.042	1.046	1.048	1.049
6	1.015	1.023	1.031	1.035	1.037	1.039
7	1.000	1.010	1.019	1.024	1.027	1.029
8	.986	.997	1.008	1.013	1.016	1.018
9	.973	.985	.997	1.002	1.006	1.008
10	.960	.973	.986	.992	.996	.998

The above table is represented graphically in Graph I of Appendix B which follows. Appendix B also contains a contour map of $c_1(\epsilon, n)$ (given in Graph II), and the tabular values used to plot Graph II (given in Table IV).

APPENDIX B

GRAPHICAL REPRESENTATION OF THE TABLE FOR e_1

GRAPH I

G_1 AS A FUNCTION OF ϵ AND n

THIS GRAPH REPRESENTS THE VALUES IN TABLE III

OF APPENDIX A

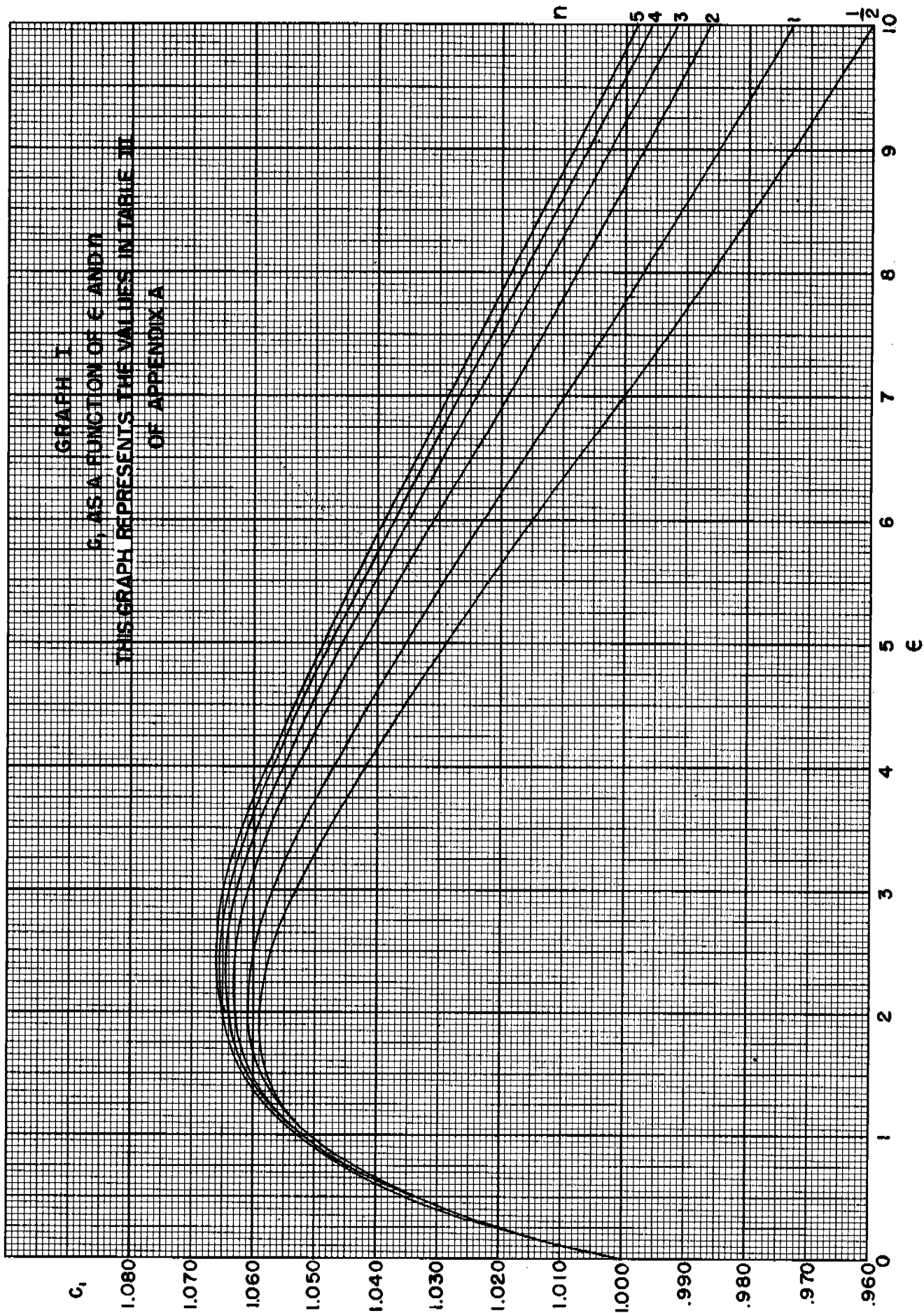


TABLE IV

In Graph II on the preceding page n is plotted as a function of ε for constant values of c_1 . In the following table the values used to plot these curves are tabulated with ε given as a function of c_1 and n .

c_1	n					
	1/2	1	2	3	4	5
1.000	0.00	0.00	0.00	0.00	0.00	0.00
1.010	0.11	0.11	0.11	0.11	0.11	0.11
1.020	0.25	0.25	0.25	0.25	0.25	0.25
1.030	0.43	0.42	0.42	0.41	0.40	0.40
1.040	0.65	0.65	0.64	0.64	0.61	0.60
1.050	0.97	0.97	0.94	0.92	0.91	0.90
1.060	----	1.69	1.50	1.44	1.40	1.35
1.063	----	----	1.98	1.75	1.63	1.60
1.065	----	----	----	----	2.00	1.92
1.065	----	----	----	----	2.73	2.98
1.063	----	----	2.45	2.95	3.18	3.35
1.060	----	2.50	3.12	3.43	3.62	3.75
1.050	3.29	3.73	4.28	4.55	4.75	4.87
1.040	4.15	4.60	5.20	5.52	5.73	5.89
1.030	4.92	5.44	6.06	6.45	6.70	6.85
1.020	5.65	6.23	6.90	7.37	7.65	7.85
1.010	6.34	6.98	7.80	8.30	8.60	8.83
1.000	7.00	7.75	8.70	9.22	9.53	9.80
.990	7.72	8.57	9.65	----	----	----
.980	8.45	9.40	----	----	----	----
.970	9.20	----	----	----	----	----

APPENDIX C

EVALUATION OF CERTAIN INTEGRALS

(The auxiliary variables used in this appendix have no connection with the notation of the rest of the report.)

1. Evaluation of $J \equiv \int_0^1 \ln \left[1 + \frac{\epsilon}{2} (1 - \mu^2) \right] d\mu$

On putting $1 + \frac{\epsilon}{2} \equiv K^2$, $\mu \equiv x (2/\epsilon)^{1/2}$, AND $(\epsilon/2)^{1/2} \equiv x_1$,

we find

$$\begin{aligned} J &= \left(\frac{2}{\epsilon}\right)^{1/2} \int_0^{x_1} \ln(K^2 - x^2) dx \\ &= \left(\frac{2}{\epsilon}\right)^{1/2} \int_0^{x_1} \ln(K+x) dx + \left(\frac{2}{\epsilon}\right)^{1/2} \int_0^{x_1} \ln(K-x) dx \\ &= \left(\frac{2}{\epsilon}\right)^{1/2} \left[(K+x) \ln(K+x) - (K+x) \right] - \left[-(K-x) \ln(K-x) + (K-x) \right] \Big|_0^{x_1} \\ &= \left(\frac{2}{\epsilon}\right)^{1/2} \left[K \ln \frac{K+x_1}{K-x_1} + x_1 \ln(K^2 - x_1^2) - 2x_1 \right] \end{aligned}$$

Now $K^2 - x_1^2 = 1$ AND $\frac{K+x_1}{K-x_1} = \frac{K^2 + x_1^2 + 2Kx_1}{K^2 - x_1^2} = 1 + \epsilon + \sqrt{\epsilon(2+\epsilon)}$

THUS $J = \left(1 + \frac{2}{\epsilon}\right)^{1/2} \ln[1 + \epsilon + \sqrt{\epsilon(2+\epsilon)}] - 2$

2. EVALUATION OF $I_{1/2} \equiv \int_0^1 [1 + b(1-x^2)]^{1/2} dx$

$$= (1+b)^{1/2} \int_0^1 \left[1 - \frac{b}{1+b} x^2 \right]^{1/2} dx$$

LET $x \left(\frac{b}{1+b}\right)^{1/2} \equiv \sin \theta$, SO THAT θ RANGES FROM 0 TO

$$\theta_m = \sin^{-1} \left(\frac{b}{1+b}\right)^{1/2} = \tan^{-1} b^{1/2}$$

THEN

$$I_{1/2} = (1+b) b^{-1/2} \int_0^{\theta_m} \cos^2 \theta d\theta$$

NOW

$$\begin{aligned} \int_0^{\theta_m} \cos^2 \theta d\theta &= \left(\frac{1}{2}\right) (\theta_m + \sin \theta_m \cos \theta_m) \\ &= \left(\frac{1}{2}\right) \left[\tan^{-1} b^{1/2} + \frac{b^{1/2}}{1+b} \right] \end{aligned}$$

THUS

$$I_{1/2} = \left(\frac{1}{2}\right) \left[1 + [1+b] b^{-1/2} \tan^{-1} b^{1/2} \right]$$

3. GIVEN $I_n \equiv \int_0^1 [1 + b(1-x^2)]^n dx$, TO DERIVE A RECURSION FORMULA FOR I_{n+1}

$$I_{n+1} = \int_0^1 [1 + b(1-x^2)]^n [1 + b(1-x^2)] dx$$

$$\begin{aligned}
I_{n+1} - (1+b)I_n &= -b \int_0^1 [1+b(1-x^2)]^n x^2 dx \\
&= 1/2 \int_0^1 x [1+b(1-x^2)]^n d[1+b(1-x^2)] \\
&= \frac{1}{2(n+1)} [1+b(1-x^2)]^{n+1} \Big|_0^1 - \frac{1}{2(n+1)} \int_0^1 [1+b(1-x^2)]^n dx \\
&= \frac{1}{2(n+1)} - \frac{I_{n+1}}{2(n+1)}
\end{aligned}$$

Thus $\frac{2n+3}{2(n+1)} I_{n+1} = (1+b)I_n + \frac{1}{2(n+1)}$

and $I_{n+1} = \frac{1}{2n+3} [2(n+1)(1+b)I_n + 1]$

We therefore have

$$I_{3/2} = (1/4) [3(1+b)I_{1/2} + 1]$$

$$I_{5/2} = (1/6) [5(1+b)I_{3/2} + 1]$$

$$I_{7/2} = (1/8) [7(1+b)I_{5/2} + 1]$$

$$I_{9/2} = (1/10) [9(1+b)I_{7/2} + 1]$$

LS
TABLE OF SYMBOLS

		Page where symbol first appears
A	uniform cross-sectional area	10
a	a parameter	9
a_0	a parameter characteristic of the Pidduck-Kent solution	10
B	a constant	9
b	$a_0/(1-a_0)$	16
c	initial mass of the powder	3
c_1	a function of ϵ and n	19
c_v	specific heat at constant volume	6
E_i	total internal energy	11
f	a function of x alone	7
f'	abbreviation for $f'(x)$	8
f_b	abbreviation for $f(x_b)$	10
H	probability integral	18
I_n	$\equiv \int_0' [1 + b(1-x^2)]^n dx$	21
J	$\equiv \int_0' \ln [1 + \frac{\epsilon}{2}(1-\mu^2)] d\mu$	16
k	a solution of $\epsilon = 2\kappa e^{\kappa} \int_0' e^{-\kappa\mu^2} d\mu$	17
m	mass of the projectile	3
m_e	effective mass	13
n	the polytropic index, $1/(\gamma - 1)$	13
p	pressure of the gas	5
p_b	pressure at the base of the projectile	3
p_e	effective pressure	13
p_o	breech pressure	3

\bar{P}	space mean pressure behind the projectile	3
Q	$\equiv \frac{1}{2n} \left[\frac{1}{b} - \frac{2(n+1)}{\epsilon} \right]$	20
R	$\equiv \int_0^1 (1 - a_0 \mu^2)^n \mu^2 d\mu$	13
R_1	the gas constant per unit mass	6
S	$\equiv \int_0^1 (1 - a_0 \mu^2)^n d\mu$	10
T	absolute temperature of the powder gas	6
t	time	5
V	velocity of the projectile	3
W_c	kinetic energy of the powder plus powder gas	3
W_p	kinetic energy of the projectile	4
W_{total}	total kinetic energy of projectile and powder	13
w	the variable $\left(\frac{1}{\rho} - \eta \right)^{-1}$	8
w_0	the variable w_0	8
x	initial distance of any gas particle from the breech	5
x_b	initial distance from the breech to the base of the projectile	10
y	distance from the breech at time t of a particle whose breech distance was initially x .	5
z	reduced breech distance of a particle	7
α	initial slope of $(2n + 3)/\delta$ versus n .	19
β	ratio (final slope/initial slope) of $(2n + 3)/\delta$ versus n	19
γ	effective ratio of specific heats	6
γ'	ratio of specific heats of the gas	6
$\delta \equiv S/R$		12
ϵ	charge-projectile mass ratio	3
η	specific covolume of the powder gas	6

μ dummy variable	10
ρ density of the gas	5
ρ_{av} average density	11
ρ_0 initial density of the gas in the special solution	5
$\sigma \equiv k^{1/2}$	18
ϕ and ψ functions of time alone	6

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